

CATEGORIFICATION OF A PARABOLIC HECKE MODULE VIA SHEAVES ON MOMENT GRAPHS

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ABSTRACT. We investigate certain categories, associated by Fiebig to the geometric representation of a Coxeter system, via sheaves on Bruhat graphs. We modify Fiebig's definition of translation functors in order to extend it to the singular setting and use it to categorify a parabolic Hecke module.

1. INTRODUCTION

A typical problem in the representation theory of Kac-Moody algebras is to understand the composition series of standard objects in the corresponding category \mathcal{O} of Bernstein, Gelfand and Gelfand ([2]). In the case of a standard object lying in a regular block, this question is the core of the Kazhdan-Lusztig theory and the answer is known to be given by the Kazhdan-Lusztig polynomials evaluated in 1. If we consider a singular block, we only have to replace these polynomials by their parabolic analogue. In the case of a principal block, this fact was conjectured by Kazhdan and Lusztig in [15] and proved in several steps in [16], [1], [5]. A fundamental role in the proof of the Kazhdan-Lusztig conjecture was played by the geometric interpretation of the problem in terms of perverse sheaves and local intersection cohomology complexes. In particular, one could study certain properties of the Hecke algebra in the category of equivariant perverse sheaves on the corresponding flag variety.

The procedure of considering a complicated object, such as a category, in order to understand an easier one is motivated by the fact that the extra structure could provide us with new tools and allow us to prove and hopefully generalise certain phenomena that are hard to address directly.

To any Coxeter system (W, \mathcal{S}) and any subset of the set of simple reflections $J \subseteq \mathcal{S}$, Deodhar associated in [6] the parabolic Hecke module \mathbf{M}^J . The aim of this paper is to give a categorification of this module, for any J generating a finite subgroup. If \mathcal{W} is a Weyl group, there is a partial flag variety Y corresponding to J , equipped with an action of a maximal torus T , and, as for the regular case, one possible categorification is given by the category of T -equivariant perverse sheaves on Y . Our goal is to describe a general categorification, which can be defined also in the case in which there is no geometry available. In order to do this, our main tools will be Bruhat moment graphs and sheaves on them. We will see how these objects come naturally into the picture.

Moment graphs appeared for the first time in [13] as 1-skeletons of “nice” actions of tori on “nice” complex algebraic varieties. In particular, Goresky, Kottwitz and MacPherson were able to describe explicitly the equivariant cohomology of these varieties only using the data encoded in the underlying moment graphs. Inspired by this result, Braden and MacPherson ([4]) could study the equivariant intersection cohomology of a complex algebraic variety equipped with a Whitney stratification, stable with respect to the torus action. In order to do so, they introduced the notion of sheaves on moment graphs and, in particular, of *canonical sheaves*. Later on, we will refer to this class of sheaves as *BMP*-sheaves.

Even if moment graphs arose originally from geometry, Fiebig observed that it is possible to give an axiomatic definition of them (cf. [11]). In particular, he associated a moment graph to

any Coxeter datum $(\mathcal{W}, \mathcal{S}, J)$ as above and, in the case of $J = \emptyset$, he used it to give an alternative construction of Soergel's category of bimodules associated to a reflection faithful representation of $(\mathcal{W}, \mathcal{S})$ (cf. [7]). The indecomposable objects of the category defined by Fiebig are precisely the *BMP*-sheaves, that, if \mathcal{W} is a Weyl group, are related to the intersection cohomology complexes, the simples in the category of perverse sheaves. A fundamental step in Fiebig's realisation of this category were translation functors, whose definition we extend to the parabolic setting (see §4.1).

The paper is organised as follows.

In Section 2 we recall the definition of the Hecke module \mathbf{M}^J and the fact that it is the unique free $\mathbb{Z}[v, v^{-1}]$ -module having rank $|\mathcal{W}/\langle J \rangle|$ and equipped with a structure of a module over the Hecke algebra \mathbf{H} , which is described in terms of the action of the \underline{H}_s 's. \underline{H}_s denotes the Kazhdan-Lusztig element corresponding to $s \in \mathcal{S}$. Then by *categorification* of \mathbf{M}^J we mean an exact category \mathcal{C} (in the sense of Quillen, [20]) together with exact functors G and $\{F_s\}_{s \in \mathcal{S}}$, that provide \mathcal{C} with the structure of a $\mathbb{Z}[v, v^{-1}]$ -module and \mathbf{H} -module, such that there exists an isomorphism from the Grothendieck group of \mathcal{C} to the parabolic module, satisfying certain compatibility conditions with these functors coming from the defining-properties of \mathbf{M}^J (see Definition 2.2).

In the third section we introduce the objects we will be dealing with in the rest of the paper. In particular, we review basic concepts of the theory of moment graphs and sheaves on them.

Section 4 is about \mathbb{Z} -graded modules over \mathcal{Z}^J , the structure algebra of a parabolic Bruhat graph. In particular, for any $s \in \mathcal{S}$, we define the translation functor ${}^s\theta$ and then the category \mathcal{H}^J of *special \mathcal{Z}^J -modules*. By definition, this category turns out to be stable under the shift in degree, that we denote by $\langle \cdot \rangle$, and under ${}^s\theta$ for all $s \in \mathcal{S}$. At this point, we are able to state our main theorem.

Theorem 4.1 *The category \mathcal{H}^J together with the shift in degree and (shifted) translation functors is a categorification of the parabolic module \mathbf{M}^J .*

The rest of the paper is devoted to the proof of this result.

In particular, in Section 5 we study certain subquotients of objects in \mathcal{H}^J and in this way are able to define the character map $h^J : [\mathcal{H}^J] \rightarrow \mathbf{M}^J$ and to show that the functors ${}^s\theta \circ \langle 1 \rangle$'s satisfy the desired compatibility condition (Proposition 5.1), while for $\langle -1 \rangle$ this is trivially fulfilled.

In the last section we prove that the character map is an isomorphism. The surjectivity follows from an easy inductive procedure (Lemma 6.1), while the proof of the injectivity is quite involved. We derive the result from the regular case, that is known to be true by [7]. The key-idea is to categorify a certain injective map $i : \mathbf{M}^J \hookrightarrow \mathbf{H}$, that is to define an exact functor $I : \mathcal{H}^J \rightarrow \mathcal{H}^\emptyset$ such that the following diagram commutes

$$\begin{array}{ccc} [\mathcal{H}^J] & \xrightarrow{[I]} & [\mathcal{H}^\emptyset] \\ h^J \downarrow & & \downarrow h^\emptyset \\ \mathbf{M}^J & \xrightarrow{i} & \mathbf{H} \end{array}$$

In order to construct and investigate the functor I , we give a realisation of \mathcal{H}^J via *BMP*-sheaves (Proposition 6.1) and then use Fiebig's idea of interchanging global and local viewpoints (cf. [7]).

2. HECKE MODULES

Here, we recall some classical constructions, such as the definition of the Hecke algebra \mathbf{H} and of its parabolic modules \mathbf{M}^J and \mathbf{N}^J , following [21]. We close the section by explaining what a categorification of the parabolic Hecke module \mathbf{M}^J is.

2.1. Hecke algebra. The Hecke algebra associated to a Coxeter system $(\mathcal{W}, \mathcal{S})$ is nothing but a quantisation of the group ring $\mathbb{Z}[\mathcal{W}]$.

In order to be more precise, we need to fix some notation. Let \leq be the Bruhat order on \mathcal{W} and $l : \mathcal{W} \rightarrow \mathbb{Z}$ be the length function associated to \mathcal{S} . Moreover, denote by $\mathcal{L} := \mathbb{Z}[v, v^{-1}]$ the ring of Laurent polynomials in the variable v over \mathbb{Z} .

Definition 2.1. *The Hecke algebra $\mathbf{H} = \mathbf{H}(\mathcal{W}, \mathcal{S})$ is the free \mathcal{L} -module having basis $\{H_x \mid x \in \mathcal{W}\}$, subject to the following relations:*

$$(1) \quad H_s H_x = \begin{cases} H_{sx} & \text{if } sx > s \\ (v^{-1} - v)H_x + H_{sx} & \text{if } sx < x \end{cases}$$

It is well known that this defines an associative \mathcal{L} -algebra (cf. [14]).

It is easy to verify that H_x is invertible for any $x \in \mathcal{W}$ and this allows us to define an involution on \mathbf{H} . This is the unique ring homomorphism $- : \mathbf{H} \rightarrow \mathbf{H}$ such that $\overline{v} = v^{-1}$ and $\overline{H_x} = (H_{x^{-1}})^{-1}$.

In [15] Kazhdan and Lusztig showed the existence of another basis for \mathbf{H} , the so-called *Kazhdan-Lusztig basis*, that they used to define complex representations of the Hecke algebra and hence of the Coxeter group. The entries of the change of basis matrix are given by a family of polynomials in $\mathbb{Z}[v]$: the *Kazhdan-Lusztig polynomials*.

2.1.1. Parabolic Hecke modules. In [6] Deodhar generalised this construction to the parabolic setting in the following way.

Let \mathcal{W}, \mathcal{S} and \mathbf{H} be as above. Now, fix a subset $J \subseteq \mathcal{S}$ and denote by $\mathcal{W}_J = \langle J \rangle$ the subgroup of \mathcal{W} generated by J . Clearly, (\mathcal{W}_J, J) is also a Coxeter system and it makes sense to consider its Hecke algebra $\mathbf{H}_J = \mathbf{H}(\mathcal{W}_J, J)$.

For any simple reflection $s \in \mathcal{S}$, consider the quadratic relation satisfied by the H_s 's: $(H_s)^2 = (v^{-1} - v)H_s + H_e$, that is $(H_s + v)(H_s - v^{-1}) = 0$. If $u \in \{v^{-1}, -v\}$, we may define a map of \mathcal{L} -modules $\varphi_u : \mathbf{H}_J \rightarrow \mathcal{L}$ by $H_s \mapsto u$. In this way \mathcal{L} gets a structure of a \mathbf{H}_J -bimodule, that we denote by $\mathcal{L}(u)$.

The parabolic Hecke modules are defined as $\mathbf{M}^J := \mathcal{L}(v^{-1}) \otimes_{\mathbf{H}_J} \mathbf{H}$ and $\mathbf{N}^J := \mathcal{L}(-v) \otimes_{\mathbf{H}_J} \mathbf{H}$. As in the regular case, it is possible to define an involutive automorphism of these modules. Namely,

$$(2) \quad \begin{array}{ccc} - : \mathcal{L}(u) \otimes_{\mathbf{H}_J} \mathbf{H} & \rightarrow & \mathcal{L}(u) \otimes_{\mathbf{H}_J} \mathbf{H} \\ a \otimes H & \mapsto & \overline{a} \otimes \overline{H} \end{array}$$

For $u \in \{v^{-1}, -v\}$, let $H_w^{J,u} := 1 \otimes H_w \in \mathcal{L}(u) \otimes_{\mathbf{H}_J} \mathbf{H}$. Denote by \mathcal{W}^J the set of minimal representatives of $\mathcal{W}/\mathcal{W}_J$.

Theorem 2.1 ([6]).

1. For all $w \in \mathcal{W}^J$ there exists a unique element $\underline{H}_w^{J,v^{-1}} \in \mathbf{M}^J$ such that:

$$\begin{aligned} \text{(i): } \underline{H}_w^{J,v^{-1}} &= \underline{H}_w^{J,v^{-1}} \\ \text{(ii): } \underline{H}_w^{J,v^{-1}} &= \sum_{y \in \mathcal{W}_J} m_{y,w}^J H_y^{J,v^{-1}}, \end{aligned}$$

where the $m_{y,w}^J$ are such that $m_{w,w}^J = 1$ and $m_{y,w}^J \in v\mathbb{Z}[v]$ if $y \neq w$.

2. For all $w \in \mathcal{W}^J$ there exists a unique element $\underline{H}_w^{J,-v} \in \mathbf{N}^J$ such that:

- (i): $\overline{H_w^{J,-v}} = \underline{H}_w^{J,-v}$
(ii): $\underline{H}_w^{J,-v} = \sum_{y \in \mathcal{W}^J} n_{y,w}^J H_y^{J,-v},$

where the $n_{y,w}^J$ are such that $n_{w,w}^J = 1$ and $n_{y,w}^J \in v\mathbb{Z}[v]$ if $y \neq w$.

Remark 2.1. In the case $J = \emptyset$, the two parabolic modules coincide with the regular module: $\mathbf{M}^\emptyset = \mathbf{N}^\emptyset = \mathbf{H}$.

From now on, we will focus on the case $u = v^{-1}$, that is we will deal only with \mathbf{M}^J . The action of the Hecke algebra \mathbf{H} on \mathbf{M}^J is the following one. Let $s \in \mathcal{S}$ be a simple reflection and let $x \in \mathcal{W}^J$, then we have (cf. [[21]§3]):

$$(3) \quad \underline{H}_s \cdot H_x^{J,v^{-1}} = \begin{cases} H_{sx}^{J,v^{-1}} + vH_x^{J,v^{-1}} & \text{if } sx \in \mathcal{W}^J, sx > x \\ H_{sx}^{J,v^{-1}} + v^{-1}H_x^{J,v^{-1}} & \text{if } sx \in \mathcal{W}^J, sx < x \\ (v + v^{-1})H_x^{J,v^{-1}} & \text{if } sx \notin \mathcal{W}^J \end{cases}$$

2.2. Definition of categorification of \mathbf{M}^J . For any exact category in the sense of Quillen [20] \mathcal{C} , let us denote by $[\mathcal{C}]$ its Grothendieck group. For an exact functor F on \mathcal{C} , let us denote by $[F]$ the induced endomorphism of $[\mathcal{C}]$. For an object C of \mathcal{C} , we denote by $[C]$ its image in the Grothendieck group $[\mathcal{C}]$. Finally, we are ready to explain which data we need to categorify the parabolic module \mathbf{M}^J .

By a *categorification* of \mathbf{M}^J , we mean an exact category \mathcal{C} together with an exact auto-functor G and a family of exact endofunctors $\{F_s\}_{s \in \mathcal{S}}$ satisfying the following requirements:

- C1:** $[\mathcal{C}]$ becomes a $\mathbb{Z}[v, v^{-1}]$ -module via $v^i \cdot [A] = [G^i A]$ for any $i \in \mathbb{Z}$ and there is an isomorphism $h^J : [\mathcal{C}] \xrightarrow{\sim} \mathbf{M}^J$ of $\mathbb{Z}[v, v^{-1}]$ -modules
C2: for any simple reflection $s \in \mathcal{S}$, we have an isomorphism of functors $GF_s \cong F_s G$ in the category
C3: for any simple reflection $s \in \mathcal{S}$, the following diagram commutes

$$\begin{array}{ccc} [\mathcal{C}] & \xrightarrow{[F_s]} & [\mathcal{C}] \\ h^J \downarrow & & \downarrow h^J \\ \mathbf{M}^J & \xrightarrow{\underline{H}_s} & \mathbf{M}^J \end{array}$$

Remark 2.2. Our notion of \mathbf{M}^J -categorification differs from the one of Mazorchuk and Stroppel (cf. [19], Remark 7.8). Indeed, we made the (weaker) requirement of \mathcal{C} being exact instead of abelian. Furthermore, we give a categorification in the case where the parabolic subgroup is finite and do not assume the Coxeter system to be finite.

The rest of the paper is devoted to the construction of this categorification. In particular, we will generalise a categorification of the Hecke algebra obtained by Fiebig in [11], which is known, by results in [7], to be equivalent to the one via Soergel's bimodules in [22].

3. SHEAVES ON MOMENT GRAPHS

3.1. Moment graphs. We recall in this section some definitions from [7], [8], [18], in order to fix the notation.

Definition 3.1 (cf.[7], §3.1). *Let k be a field and let V be a finite dimensional k -vector space. A V -moment graph is given by $(\mathcal{V}, \mathcal{E}, \leq, l)$, where:*

- (MG1) $(\mathcal{V}, \mathcal{E})$ is a directed graph without directed cycles nor multiple edges
- (MG2) \leq is a partial order on \mathcal{V} such that if $x, y \in \mathcal{V}$ and $E : x \rightarrow y \in \mathcal{E}$, then $x \leq y$
- (MG3) $l : \mathcal{E} \rightarrow \mathbb{P}^1(V)$ is a map called the label function.

Since a V -moment graph is an ordered graph, whose edges are labeled by one-dimensional subspaces of V , a morphism of two such objects is be given by a morphism of oriented graphs together with a family of certain automorphisms of V .

Definition 3.2 (cf.[18]). *A morphism between two V -moment graphs*

$$f : (\mathcal{V}, \mathcal{E}, \leq, l) \rightarrow (\mathcal{V}', \mathcal{E}', \leq', l')$$

is given by $(f_{\mathcal{V}}, \{f_{l,x}\}_{x \in \mathcal{V}})$, where

- (MORPH1) $f_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{V}'$ is any map of posets such that, if $x \rightarrow y \in \mathcal{E}$, then either $f_{\mathcal{V}}(x) \rightarrow f_{\mathcal{V}}(y) \in \mathcal{E}'$, or $f_{\mathcal{V}}(x) = f_{\mathcal{V}}(y)$.

For an edge $E : x \rightarrow y \in \mathcal{E}$ such that $f_{\mathcal{V}}(x) \neq f_{\mathcal{V}}(y)$, let $f_{\mathcal{E}}(E) := f_{\mathcal{V}}(x) \rightarrow f_{\mathcal{V}}(y)$.

- (MORPH2) *For all $x \in \mathcal{V}$, $f_{l,x} : V \rightarrow V \in \text{Aut}_k(V)$ is such that, if $E : x \rightarrow y \in \mathcal{E}$ and $f_{\mathcal{V}}(x) \neq f_{\mathcal{V}}(y)$, the following two conditions are verified.*

- (MORPH2a) $f_{l,x}(l(E)) = l'(f_{\mathcal{E}}(E))$.

- (MORPH2b) $\pi \circ f_{l,x} = \pi \circ f_{l,y}$, where π is the canonical quotient map $\pi : V \rightarrow V/(\beta V)$.

If $f : \mathcal{G} = (\mathcal{V}, \mathcal{E}, \leq, l) \rightarrow \mathcal{G}' = (\mathcal{V}', \mathcal{E}', \leq', l')$ and $g : \mathcal{G}' \rightarrow \mathcal{G}'' = (\mathcal{V}'', \mathcal{E}'', \leq'', l'')$ are two morphisms of k -moment graphs, then there is a natural way to define the composition. Namely, $g \circ f := (g_{\mathcal{V}'} \circ f_{\mathcal{V}}, \{g_{l', f_{\mathcal{V}}(x)} \circ f_{l,x}\}_{x \in \mathcal{V}})$.

It is now not hard to see that the composition of two morphisms between k -moment graphs is again a morphism, and it is associative. Thus it makes sense to give the following definition.

Definition 3.3. *We denote by $\mathbf{MG}(V)$ the category of V -moment graphs corresponding morphisms.*

3.1.1. *Bruhat graphs.* Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system and denote by \mathcal{T} the set of reflections, that is

$$\mathcal{T} = \{ws w^{-1} | s \in \mathcal{S}; w \in \mathcal{W}\}$$

Let V be the geometric representation of $(\mathcal{W}, \mathcal{S})$ (cf. [14], 5.3). Then V has a basis indexed by the set of simple reflections $\Pi = \{\alpha_s\}$ and s acts on it in the following way

$$s : v \mapsto v - 2\langle v, \alpha_s \rangle \alpha_s$$

where $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ denotes the symmetric bilinear form making compatible the angle between any pair of elements of the basis and the order of their product.

Let us consider a subset $J \subseteq \mathcal{S}$ and keep the same notation as in the previous section. Chose $\lambda \in V$ such that $\mathcal{W}_J = \text{Stab}_{\mathcal{W}}(\lambda)$. Then \mathcal{W}^J can be identify with the orbit $\mathcal{W} \cdot \lambda$ via $x \mapsto x(\lambda)$.

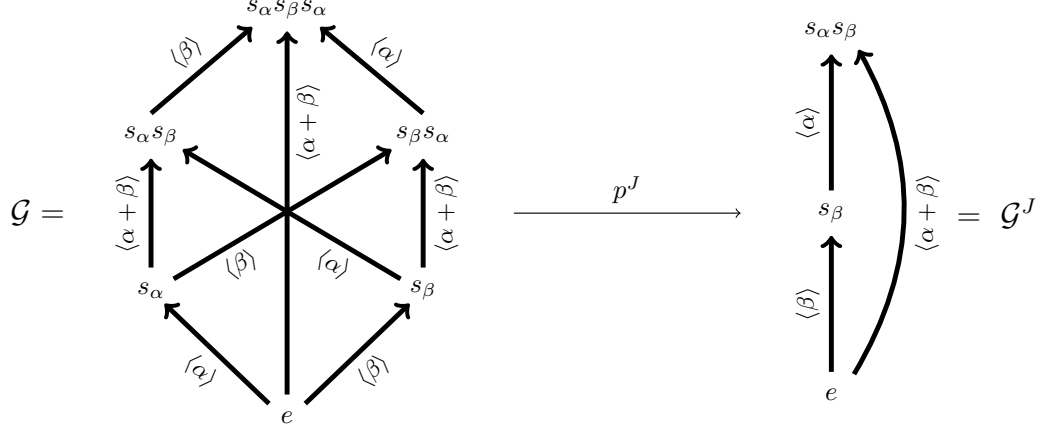
Definition 3.4 (cf.[7], §2.2). *The Bruhat moment graph \mathcal{G}^J associated to the Coxeter datum $(\mathcal{W}, \mathcal{S}, J)$ is a V -moment graph, whose set of vertices is given by $\mathcal{W} \cdot \lambda \leftrightarrow \mathcal{W}^J$, equipped with the (induced) Bruhat order. We connect $x, y \in \mathcal{W}^J$ if and only if there exists a reflection $t \in \mathcal{T}$ such that $x(\lambda) = ty(\lambda)$ and $x \neq y$, that is $y = txw$, for some $w \in \mathcal{W}_J$, and $y \notin x\mathcal{W}_J$. Moreover the edge is oriented from x towards y if $\ell(x) < \ell(y)$. Finally, the label $l(x \rightarrow txw)$ is given by the line generated by $x(\lambda) - tx(\lambda)$ in $\mathbb{P}(V)$.*

Consider now two Bruhat moment graphs on V , the geometric representation of \mathcal{W} : $\mathcal{G} = \mathcal{G}(\mathcal{W}, \emptyset)$ and $\mathcal{G}^J = \mathcal{G}(\mathcal{W}, J)$. The canonical quotient map $p^J : \mathcal{G} \rightarrow \mathcal{G}^J$ is given by $(p_{\mathcal{V}}^J, \{p_{l,x}^J\})$, where

- $p_V^J : x \rightarrow x^J$, with x^J minimal length representative of the coset $x\mathcal{W}_J$
- $p_{l,x}^J = \text{Id}_{V^*}$ for any $x \in \mathcal{W}$

It is clear that p^J is a well-defined morphism of V -moment graphs.

Example 3.1. Let $\mathcal{W} = S_3$, the symmetric group on three letters. Then in this case $V = \mathbb{R}^2$, $\Pi = \{\alpha, \beta\}$ and the angle between the two roots is $\frac{\pi}{3}$. Let us fix $J = \{s_\alpha\}$, then p^J is as follows.



We have $p_V^J(e) = p_V^J(s_\alpha) = e$, $p_V^J(s_\beta) = p_V^J(s_\beta s_\alpha) = s_\beta$ and $p_V^J(s_\alpha s_\beta) = p_V^J(s_\alpha s_\beta s_\alpha) = s_\alpha s_\beta$. It is immediate to verify that $p_{l,x} = \text{Id}_{\mathbb{R}^2}$ satisfies (MORPH2a) and (MORPH2b) for any $x \in S_3$.

3.2. Sheaves on a V -moment graph.

3.2.1. Conventions. For any finite dimensional vector space V over the field k (with $\text{char } k \neq 2$), we denote by $S = \text{Sym}(V)$ its symmetric algebra. S is a polynomial ring and we provide it with the grading induced by setting $S_{\{2\}} = V$. From now on, all the S -modules will be finitely generated and \mathbb{Z} -graded. Moreover, we will consider only degree zero morphisms between them. Finally, for a graded S -module $M = \bigoplus_i M_{\{i\}}$ and for $j \in \mathbb{Z}$, we denote by $M\langle j \rangle$ the \mathbb{Z} -graded S -module obtained from M by shifting the grading by j , that is $(M\langle j \rangle)_{\{i\}} = M_{\{j+i\}}$.

Definition 3.5 ([4]). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \mathbf{MG}(V)$, then a sheaf \mathcal{F} on \mathcal{G} is given by $(\{\mathcal{F}^x\}, \{\mathcal{F}^E\}, \{\rho_{x,E}\})$, where

- (SH1) for all $x \in \mathcal{V}$, \mathcal{F}^x is an S -module;
- (SH2) for all $E \in \mathcal{E}$, \mathcal{F}^E is an S -module such that $l(E) \cdot \mathcal{F}^E = \{0\}$;
- (SH3) for $x \in \mathcal{V}$, $E \in \mathcal{E}$, $\rho_{x,E} : \mathcal{F}^x \rightarrow \mathcal{F}^E$ is a homomorphism of S -modules defined if x lies on the border of the edge E .

Remark 3.1. We may consider the following topology on \mathcal{G} (cf. [4], §1.3). We say that a subgraph \mathcal{G}' of \mathcal{G} is open, if whenever a vertex x is in \mathcal{G}' , then all edges adjacent to x are in \mathcal{G}' . With this topology, the object we defined above is actually a proper sheaf of S -modules on \mathcal{G} . For our purposes it will be sufficient to consider the sheaves as purely combinatorial and algebraic objects.

Example 3.2 (cf. [4], §1). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \mathbf{MG}(V)$, then its structure sheaf \mathcal{Z} is given by

- for all $x \in \mathcal{V}$, $\mathcal{Z}^x = S$
- for all $E \in \mathcal{E}$, $\mathcal{Z}^E = S/l(E) \cdot S$

- for all $x \in \mathcal{V}$ and $E \in \mathcal{E}$, such that x is in the border of the edge E , $\rho_{x,E} : S \rightarrow S/l(E) \cdot S$ is the canonical quotient map

Definition 3.6 ([9]). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \mathbf{MG}(V)$ and let $\mathcal{F} = (\{\mathcal{F}^x\}, \{\mathcal{F}^E\}, \{\rho_{x,E}\})$, $\mathcal{F}' = (\{\mathcal{F}'^x\}, \{\mathcal{F}'^E\}, \{\rho'_{x,E}\})$ be two sheaves on it. A morphism $\varphi : \mathcal{F} \rightarrow \mathcal{F}'$ is given by the following data

- (i) for all $x \in \mathcal{V}$, $\varphi^x : \mathcal{F}^x \rightarrow \mathcal{F}'^x$ is a homomorphism of S -modules
- (ii) for all $E \in \mathcal{E}$, $\varphi^E : \mathcal{F}^E \rightarrow \mathcal{F}'^E$ is a homomorphism of S -modules such that, for any $x \in \mathcal{V}$ on the border of $E \in \mathcal{E}$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}^x & \xrightarrow{\rho_{x,E}} & \mathcal{F}^E \\ \downarrow \varphi^x & & \downarrow \varphi^E \\ \mathcal{F}'^x & \xrightarrow{\rho'_{x,E}} & \mathcal{F}'^E \end{array}$$

Definition 3.7. Let $\mathcal{G} \in \mathbf{MG}(V)$. We denote by $\mathbf{Sh}(\mathcal{G})$ the category of sheaves on \mathcal{G} and corresponding morphisms.

3.3. Pullback of sheaves. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l)$, resp. $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \trianglelefteq', l')$, be a moment graph on Y and fix $f : \mathcal{G} \rightarrow \mathcal{G}'$ a k -momorphism of moment graphs.

Definition 3.8. Let $\mathcal{F} \in \text{Ob}(\mathbf{Sh}_{\mathcal{G}'}^k)$, then $f^*\mathcal{F} \in \text{Ob}(\mathbf{Sh}_{\mathcal{G}}^k)$ is defined as follows

(PULL1) for all $x \in \mathcal{V}$, $(f^*\mathcal{F})^x := \mathcal{F}^{f_{\mathcal{V}}(x)}$ and $s \in S$ acts on it via $f_{l,x}(s)$

(PULL2) for all $E : x \rightarrow y \in \mathcal{E}$

$$(f^*\mathcal{F})^E = \begin{cases} \mathcal{F}^{f_{\mathcal{V}}(x)}/l(E)\mathcal{F}^{f_{\mathcal{V}}(x)} & \text{if } f_{\mathcal{V}}(x) = f_{\mathcal{V}}(y) \\ \mathcal{F}^{f_{\mathcal{E}}(E)} & \text{otherwise} \end{cases}$$

and of $s \in S$ acts on $(f^*\mathcal{F})^E$ via $f_{l,x}(s)$.

(PULL3) for all $x \in \mathcal{V}$ and $E \in \mathcal{E}'$, such that $E : x \rightarrow y$,

$$(f^*\rho)_{x,E} = \begin{cases} \text{canonical quotient map} & \text{if } f_{\mathcal{V}}(x) = f_{\mathcal{V}}(y) \\ \rho_{f_{\mathcal{V}}(x), f_{\mathcal{E}}(E)} & \text{otherwise} \end{cases}$$

Remark 3.2. For all $E \in \mathcal{E}$, the action of S on $(f^*\mathcal{F})^E$ in (PULL2) is well-defined thanks to conditions (MORPH2a) and (MORPH2b).

We say that $f^*\mathcal{F}$ is the pullback of \mathcal{F} .

3.4. Sections of a sheaf on a moment graph. Even if $\mathbf{Sh}(\mathcal{G})$ is not a category of sheaves in the topological meaning, we may define, following [8], the notion of sections.

Definition 3.9. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \mathbf{MG}(V)$, $\mathcal{F} = (\{\mathcal{F}^x\}, \{\mathcal{F}^E\}, \{\rho_{x,E}\}) \in \mathbf{Sh}(\mathcal{G})$ and $\mathcal{I} \subseteq \mathcal{V}$. Then the set of sections of \mathcal{F} over \mathcal{I} is denoted by $\Gamma(\mathcal{I}, \mathcal{F})$ and defined as

$$\Gamma(\mathcal{I}, \mathcal{F}) := \left\{ (m_x) \in \prod_{x \in \mathcal{I}} \mathcal{F}^x \mid \begin{array}{l} \rho_{x,E}(m_x) = \rho_{y,E}(m_y) \\ \forall E : x \rightarrow y \in \mathcal{E}, x, y \in \mathcal{I} \end{array} \right\}.$$

We set $\Gamma(\mathcal{F}) := \Gamma(\mathcal{V}, \mathcal{F})$, that is the set of global sections of \mathcal{F} .

Example 3.3. A very important example is given by the set of global sections of the structure sheaf \mathcal{Z} (cf. Ex. 3.2). In this case, we get the structure algebra:

$$\mathcal{Z} := \Gamma(\mathcal{Z}) = \left\{ (z_x)_{x \in \mathcal{V}} \in \bigoplus_{x \in \mathcal{V}} S \mid \forall E : x \longrightarrow y \in \mathcal{E} \quad z_x - z_y \in l(E) \cdot S \right\}$$

It is easy to check that \mathcal{Z} , equipped with componentwise addition and multiplication, is an algebra and that there is an action of S on it by diagonal multiplication. Moreover, for any sheaf $\mathcal{F} \in \mathbf{Sh}(\mathcal{G})$, \mathcal{Z} acts on the space $\Gamma(\mathcal{F})$ via componentwise multiplication.

Let us denote by $\mathcal{Z}\text{-mod}^f$ the category of \mathbb{Z} -graded \mathcal{Z} -modules that are torsion free and finitely generated over S . Then Γ defines a functor

$$(4) \quad \Gamma : \mathbf{Sh}(\mathcal{G}) \rightarrow \mathcal{Z}\text{-mod}^f$$

3.5. BMP-sheaves. So far we have only seen one example of a sheaf on a moment graph: the structure sheaf \mathcal{Z} . We are now going to recall the notion of *BMP-sheaves*. It is more complicated than the one in Example 3.2 and indeed some notation is needed.

Let us fix $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l) \in \mathbf{MG}(V)$. For all $\mathcal{F} \in \mathbf{Sh}(\mathcal{G})$ and $x \in \mathcal{V}$, we set

$$\mathcal{E}_{\delta x} := \{E \in \mathcal{E} \mid E : x \rightarrow y\}$$

$$\mathcal{V}_{\delta x} := \{y \in \mathcal{V} \mid \exists E \in \mathcal{E}_{\delta x} \text{ such that } E : x \rightarrow y\}.$$

Moreover, for any $x \in \mathcal{V}$ let us denote $\{\triangleright x\} = \{y \in \mathcal{V} \mid y \triangleright x\}$ and define $\mathcal{F}^{\delta x}$ as the image of $\Gamma(\{\triangleright x\}, \mathcal{F})$ under the composition of the following functions:

$$u_x : \Gamma(\{\triangleright x\}, \mathcal{F}) \longrightarrow \bigoplus_{y \triangleright x} \mathcal{F}^y \longrightarrow \bigoplus_{y \in \mathcal{V}_{\delta x}} \mathcal{F}^y \xrightarrow{\oplus \rho_{y,E}} \bigoplus_{E \in \mathcal{E}_{\delta x}} \mathcal{F}^E$$

Theorem 3.1 ([4]). *Let $\mathcal{G} \in \mathbf{MG}(V)$ and let $w \in \mathcal{V}$. There exists exactly one (up to isomorphism) indecomposable sheaf $\mathcal{B}(w)$ on \mathcal{G} with the following properties:*

- (i) *If $x \in \mathcal{V}$, then $\mathcal{B}(w)^x \cong 0$, unless $x \triangleright w$. Moreover, $\mathcal{B}(w)^w \cong S$*
- (ii) *If $x, y \in \mathcal{V}$, $E : x \rightarrow y \in \mathcal{E}$, then the map $\rho_{y,E} : \mathcal{B}(w)^y \rightarrow \mathcal{B}(w)^E$ is surjective with kernel $l(E) \cdot \mathcal{B}(w)^y$*
- (iii) *If $x, y \in \mathcal{V}$, $x \triangleright y$ and $E : x \rightarrow y \in \mathcal{E}$, then $\rho_{\delta x} := \bigoplus_{E \in \mathcal{E}_{\delta x}} \rho_{x,E} : \mathcal{B}(w)^x \rightarrow \mathcal{B}(w)^{\delta x}$ is a projective cover in the category of graded S -modules.*

Later on we will refer to $\mathcal{B}(w)$ as the *BMP-sheaf*.

4. MODULES OVER THE STRUCTURE ALGEBRA OF A BRUHAT MOMENT GRAPH

Let \mathcal{Z} be the structure algebra (see §3.4) of a regular Bruhat graph $\mathcal{G} = \mathcal{G}(\mathcal{W}, \emptyset)$. In [7], Fiebig defined translation functors on the category $\mathcal{Z}\text{-mod}^f$. Using these, he defined inductively a full subcategory \mathcal{H} of $\mathcal{Z}\text{-mod}$ and proved that \mathcal{H} , in characteristic zero, is equivalent to a category of bimodules introduced by Soergel in [22]. In [11] it is shown that \mathcal{H} categorifies the Hecke algebra \mathbf{H} (and the periodic module \mathbf{M}), using translation functors. The aim of this chapter is to define translation functors in the parabolic setting and to extend some results of [11].

Let \mathcal{W} be a Weyl group, let \mathcal{S} be its set of simple reflections and let $J \subseteq \mathcal{S}$. Hereafter we will keep the notation we used in §2. Recall that, for any $z \in \mathcal{W}$, there is a unique factorisation $x = x^J x_J$, with $x^J \in \mathcal{W}^J$, $x_J \in \mathcal{W}_J$ and $l(x) = l(x^J) + l(x_J)$ (cf. [3], Proposition 2.4.4).

In [7], for all $s \in \mathcal{S}$, an involutive automorphism σ_s of the structure algebra of a regular Bruhat graph is given. In a similar way, we will define an involution ${}_s\sigma$ for a fixed simple reflection $s \in \mathcal{S}$ on the structure algebra \mathcal{Z}^J of the parabolic Bruhat moment graph \mathcal{G}^J .

Let $x, y \in \mathcal{W}^J$. Notice that $l(x \text{ --- } y) = \alpha_t$ if and only if $l(sx \text{ --- } sy) = s(\alpha_t)$, because $sxw(sy)^{-1} = sxwy^{-1}s = sts$, for some $w \in \mathcal{W}_J$.

Denote by τ_s the automorphism of the symmetric algebra S induced by the mapping $\lambda \mapsto s(\lambda)$ for all $\lambda \in V$. For any $(z_x)_{x \in \mathcal{W}^J} \in \mathcal{Z}^J$, we set ${}_s\sigma((z_x)_{x \in \mathcal{W}^J}) = (z'_x)_{x \in \mathcal{W}^J}$, where $z'_x := \tau_s(z_{(sx)^J})$. This is again an element of the structure algebra from what we have observed above.

Let us fix the following notation.

- ${}^s\mathcal{Z}^J$ is the space of invariants with respect to ${}_s\sigma$
- ${}^{-s}\mathcal{Z}^J$ is the space of anti-invariants with respect to ${}_s\sigma$

We denote moreover by $\overline{\alpha_s}$ the element of \mathcal{Z}^J whose components are all equal to α_s . We obtain the following decomposition of \mathcal{Z}^J as a ${}^s\mathcal{Z}^J$ -module.

Lemma 4.1. $\mathcal{Z}^J = {}^s\mathcal{Z}^J \oplus \overline{\alpha_s} \cdot {}^s\mathcal{Z}^J$.

Proof. (We follow [7], Lemma 5.1). Because ${}_s\sigma$ is an involution, we get $\mathcal{Z}^J = {}^s\mathcal{Z}^J \oplus {}^{-s}\mathcal{Z}^J$. Since $\overline{\alpha_s} \in \mathcal{Z}^J$ and $s(\alpha_s) = -\alpha_s$, it follows ${}_s\sigma(\overline{\alpha_s}) = -\overline{\alpha_s}$ and so $\overline{\alpha_s} \cdot {}^s\mathcal{Z}^J \subseteq {}^{-s}\mathcal{Z}^J$ and we now have to prove the other inclusion, that is every element $z \in {}^{-s}\mathcal{Z}^J$ is divisible by $\overline{\alpha_s}$ in ${}^{-s}\mathcal{Z}^J$.

If $z = (z_x) \in {}^{-s}\mathcal{Z}^J$, then, for all $x \in \mathcal{W}^J$,

$$z_x = -\tau_s(z_{(sx)^J}) \equiv -z_{(sx)^J} \pmod{\alpha_s}$$

But on the other hand,

$$z_x \equiv z_{(sx)^J} \pmod{\alpha_s}$$

It follows that $2z_x \equiv 0 \pmod{\alpha_s}$, that is α_s divides z_x in S , as $\text{char}(k) \neq 2$.

We have now to verify that $z' := \overline{\alpha_s}^{-1} \cdot z \in \mathcal{Z}$, that is $z'_x - z'_{(tx)^J} \equiv 0 \pmod{\alpha_t}$ for any $x \in \mathcal{W}^J$ and $t \in \mathcal{T}$. If $(tx)^J = (sx)^J$, there is nothing to prove; on the other hand, if $(tx)^J \neq (sx)^J$, we get the following.

$$\alpha_s \cdot (z'_x - z'_{(tx)^J}) = z_x - z_{(tx)^J} \equiv 0 \pmod{\alpha_t}$$

Since α_s and α_t are linearly independent, $\alpha_s \not\equiv 0 \pmod{\alpha_t}$ and we obtain $z'_x - z'_{(tx)^J} \equiv 0 \pmod{\alpha_t}$. \square

4.1. Translation functors and special modules. In order to define translation functors, we need an action of S on ${}^s\mathcal{Z}^J$ and \mathcal{Z}^J .

Lemma 4.2. For any $\lambda \in V$ and any $x \in \mathcal{W}^J$, let us set

$$(5) \quad c(\lambda)_x^J := \sum_{x_J \in \mathcal{W}_J} xx_J(\lambda).$$

Then $c(\lambda)^J := (c(\lambda)_x^J)_{x \in \mathcal{W}^J} \in {}^s\mathcal{Z}^J$.

Proof. It is clear that, if $c(\lambda)^J \in \mathcal{Z}^J$, then it is invariant. So we only have to prove that $c(\lambda)^J \in \mathcal{Z}^J$, that is $c(\lambda)_x^J - c(\lambda)_{(tx)^J}^J \equiv 0 \pmod{\alpha_t}$. Since for any x_J there exists an element y_J such that $xx_J = t(tx)^J y_J$, we obtain

$$\begin{aligned} \sum_{x_J \in \mathcal{W}_J} xx_J(\lambda) - \sum_{x_J \in \mathcal{W}_J} (tx)^J x_J(\lambda) &= \sum_{y_J \in \mathcal{W}_J} t(tx)^J y_J(\lambda) - \sum_{y_J \in \mathcal{W}_J} (tx)^J y_J(\lambda) \\ &= t \left(\sum_{y_J \in \mathcal{W}_J} (tx)^J y_J(\lambda) \right) - \sum_{y_J \in \mathcal{W}_J} (tx)^J y_J(\lambda) \\ &= \left(\sum_{y_J \in \mathcal{W}_J} 2 \langle (tx)^J y_J(\lambda), \alpha_t \rangle \right) \alpha_t \\ &\equiv 0 \pmod{\alpha_t} \end{aligned}$$

\square

For any $x \in \mathcal{W}^J$, denote by η_x the automorphism of the symmetric algebra S induced by the map $\lambda \mapsto c(\lambda)_x^J$ for all $\lambda \in V$. Now, by Lemma 4.2, the action of S on \mathcal{Z}^J given by

$$(6) \quad p.(z_x)_{x \in \mathcal{W}^J} = (\eta_x(p)z_x) \quad p \in S, \quad z \in \mathcal{Z}^J,$$

preserves ${}^s\mathcal{Z}^J$. Thus any \mathcal{Z}^J -module or ${}^s\mathcal{Z}^J$ -module has an S -module structure as well. Let $\mathcal{Z}^J\text{-mod}^f$, resp. ${}^s\mathcal{Z}^J\text{-mod}^f$, be the category of \mathbb{Z} -graded \mathcal{Z}^J -modules, resp. ${}^s\mathcal{Z}^J$ -modules, that are torsion free and finitely generated over S .

The *translation on the wall* is the functor ${}^{s, \text{on}}\theta : \mathcal{Z}^J\text{-mod} \rightarrow {}^s\mathcal{Z}^J\text{-mod}$ defined by the mapping $M \mapsto \text{Res}_{\mathcal{Z}^J}^{s\mathcal{Z}^J} M$.

The *translation out of the wall* is the functor ${}^{s, \text{out}}\theta : {}^s\mathcal{Z}^J\text{-mod} \rightarrow \mathcal{Z}^J\text{-mod}$ defined by the mapping $N \mapsto \text{Ind}_{\mathcal{Z}^J}^{s\mathcal{Z}^J} N$. Observe that this functor is well-defined due to Lemma 4.1.

By composition, we get a functor ${}^s\theta^J := {}^{s, \text{out}}\theta \circ {}^{s, \text{on}}\theta : \mathcal{Z}^J\text{-mod} \rightarrow \mathcal{Z}^J\text{-mod}$ that we call *(left) translation functor*.

The following proposition describes the first properties of ${}^s\theta$.

Proposition 4.1. (1): *The functors from ${}^s\mathcal{Z}^J\text{-mod}$ to ${}^s\mathcal{Z}\text{-mod}$ mapping $M \mapsto \mathcal{Z}^J\{2\} \otimes_{{}^s\mathcal{Z}^J} M$ and $M \mapsto \text{Hom}_{{}^s\mathcal{Z}^J}(\mathcal{Z}^J, M)$ are naturally equivalent.*
 (2): *The functor ${}^s\theta = \mathcal{Z}^J \otimes_{{}^s\mathcal{Z}^J} - : \mathcal{Z}^J\text{-mod} \rightarrow \mathcal{Z}^J\text{-mod}$ is selfadjoint up to a shift.*

Proof. (cf. [22], Proposition 5.10, and [7], Proposition. 5.2) By Lemma 4.1, $\{\bar{1}, \bar{\alpha}_s\}$ is a ${}^s\mathcal{Z}^J$ -basis for \mathcal{Z}^J . Let $\bar{1}^*, \bar{\alpha}_s^* \in \text{Hom}_{{}^s\mathcal{Z}^J}(\mathcal{Z}^J, {}^s\mathcal{Z}^J)$ a ${}^s\mathcal{Z}^J$ -basis dual to $\bar{1}$ and $\bar{\alpha}_s$. We have an isomorphism of ${}^s\mathcal{Z}^J$ -modules $\mathcal{Z}^J\{2\} \cong \text{Hom}_{{}^s\mathcal{Z}^J}(\mathcal{Z}^J, {}^s\mathcal{Z}^J)$ defined by the mapping $1 \mapsto \bar{\alpha}_s^*$ and $\bar{\alpha}_s \mapsto 1^*$, since $\deg(1) - 2 = -2 = \deg(\bar{\alpha}_s^*)$ and $\deg(\bar{\alpha}_s) - 2 = 0 = \deg \bar{1}^*$. Now statement (1) follows from the fact that \mathcal{Z}^J is of finite rank over ${}^s\mathcal{Z}^J$ and so $\text{Hom}_{{}^s\mathcal{Z}^J}(\mathcal{Z}^J, -) = \text{Hom}_{{}^s\mathcal{Z}^J}(\mathcal{Z}^J, {}^s\mathcal{Z}^J) \otimes_{{}^s\mathcal{Z}^J} -$.

Now the second claim follows easily, since $\mathcal{Z}^J \otimes_{{}^s\mathcal{Z}^J} -$ and $\text{Hom}_{{}^s\mathcal{Z}^J}(\mathcal{Z}^J, -)$ are, resp., left and right adjoint to the restriction functor. □

Using the selfadjointness of ${}^s\theta$ we get the following corollary

Corollary 4.1. ${}^s\theta : \mathcal{Z}^J\text{-mod} \rightarrow \mathcal{Z}^J\text{-mod}$ is exact.

4.2. Parabolic special modules. As in [7], we define, inductively, a full subcategory of $\mathcal{Z}^J\text{-mod}$.

Let $B_e^J \in \mathcal{Z}^J\text{-mod}$ be the free S -module of rank one on which $z = (z_x)_{x \in \mathcal{W}^J}$ acts via multiplication by z_e .

Definition 4.1.

- (i) *The category of special \mathcal{Z}^J -modules is the full subcategory \mathcal{H}^J of $\mathcal{Z}^J\text{-mod}^f$ whose objects are isomorphic to a direct summand of a direct sum of modules of the form $s_{i_1}\theta \circ \dots \circ s_{i_r}\theta(B_e^J)\langle n \rangle$, where $s_{i_1}, \dots, s_{i_r} \in \mathcal{S}$ and $n \in \mathbb{Z}$.*
- (ii) *The category of special ${}^s\mathcal{Z}^J$ -modules is the full subcategory ${}^s\mathcal{H}^J$ of ${}^s\mathcal{Z}^J\text{-mod}^f$ whose objects are isomorphic to a direct summand of ${}^{s, \text{on}}\theta(M)$ for some $M \in \mathcal{H}^J$.*

We are now able to state the main result of this paper:

Theorem 4.1. *The category \mathcal{H}^J together with the shift in degree and (shifted) translation functors is a categorification of the parabolic module \mathbf{M}^J .*

The rest of the paper is devoted to the proof of it.

4.3. Finiteness of special modules. Let Ω be a finite subset of \mathcal{W}^J . Then, we set

$$\mathcal{Z}^J(\Omega) := \left\{ (z_x) \in \prod_{x \in \Omega} S_k \mid \begin{array}{l} z_x \equiv z_y \pmod{\alpha_t^\vee} \\ \text{if } \exists w \in \mathcal{W}_J \text{ s.t. } y w x^{-1} = t \in \mathcal{T} \end{array} \right\}$$

If $\Omega \subseteq \mathcal{W}^J$ is s -invariant, that is $s\Omega = \Omega$, we may restrict ${}_s\sigma$ to it. We denote by ${}^s\mathcal{Z}^J(\Omega) \subseteq \mathcal{Z}^J(\Omega)$ the space of invariants and, using Lemma 4.1, we get a decomposition $\mathcal{Z}^J(\Omega) = {}^s\mathcal{Z}^J(\Omega) \oplus \overline{\alpha_s^\vee}$.

In the following lemma we prove, the *finiteness* of the special \mathcal{Z}^J -modules, as Fiebig does in [11] for special \mathcal{Z} -modules.

Lemma 4.3.

- (i) Let $M \in \mathcal{H}^J$. Then there exists a finite subset $\Omega \subset \mathcal{W}^J$ and an action of $\mathcal{Z}^J(\Omega)$ such that \mathcal{Z}^J acts on M via the canonical map $\mathcal{Z}^J \rightarrow \mathcal{Z}^J(\Omega)$.
- (ii) Let $s \in \mathcal{S}$ and let N be an object in ${}^s\mathcal{H}^J$. Then there exists a finite s -invariant subset $\Omega \subset \mathcal{W}^J$ and an action of ${}^s\mathcal{Z}^J(\Omega)$ on N such that ${}^s\mathcal{Z}^J$ acts on N via the canonical map ${}^s\mathcal{Z}^J \rightarrow {}^s\mathcal{Z}^J(\Omega)$.

Proof. (we follow [11]) We prove (i) by induction. It holds clearly for B_e , since \mathcal{Z}^J acts on it via the map $\mathcal{Z}^J \rightarrow \mathcal{Z}^J(\{e\})$. Now we have to show that if the claim is true for $M \in \mathcal{H}^J$, then it holds also for ${}^s\theta(M)$. Suppose \mathcal{Z}^J acts via the map $\mathcal{Z}^J \rightarrow \mathcal{Z}^J(\Omega)$ over M . Observe that we may assume Ω s -invariant, since we can just replace it by $\Omega \cup s\Omega$, which is still finite. In this way the ${}^s\mathcal{Z}^J$ -action on ${}^s\theta M$ via ${}^s\mathcal{Z}^J \rightarrow {}^s\mathcal{Z}^J(\Omega)$ and so we obtain ${}^s\theta M := \mathcal{Z}^J \otimes_{{}^s\mathcal{Z}^J} M = \mathcal{Z}^J(\Omega) \otimes_{{}^s\mathcal{Z}^J(\Omega)} M$.

Claim (ii) follows directly from claim (1). \square

5. MODULES OVER THE STRUCTURE ALGEBRA

We recall some notation from [8]. Let Q be the quotient field of S and, for any $M \in \mathcal{Z} - \text{mod}^f$, we denote $M_Q := M \otimes_S Q$. If \mathcal{G} is quasi-finite (see §3.1 of [8]), There is a decomposition $M_Q := M \cap \bigoplus_{x \in \mathcal{V}} M_Q^x$ and so a canonical inclusion $M \subseteq \bigoplus_{x \in \mathcal{V}} M_Q^x$. For all subsets of the set of vertices $\Omega \subseteq \mathcal{V}$, we may define:

$$M_\Omega := M \cap \bigoplus_{x \in \Omega} M_Q^x,$$

$$M^\Omega := M/M_{\mathcal{V} \setminus \Omega} = \text{im} \left(M \rightarrow M_Q = \bigoplus_{x \in \Omega} M_Q^x \right).$$

For all $x \in \mathcal{V}$, we set

$$M_{[x]} := \ker \left(M^{\{\triangleright x\}} \rightarrow M^{\{\triangleright x\}} \right)$$

If $x \triangleleft y$ and $[x, y] = \{x, y\}$, we denote

$$M_{[x, y]} := \ker \left(M^{\{\triangleright x\}} \rightarrow M^{\{\triangleright x\} \setminus \{x, y\}} \right)$$

5.1. Modules with a Verma flag. As in [8], we denote by \mathcal{V} the full subcategory of $\mathcal{Z} - \text{mod}^f$ whose objects admit a Verma flag, that is $M \in \mathcal{V}$ if and only if M^Ω is a graded free S -module for any $\Omega \subseteq \mathcal{V}$ upwardly closed with respect to the partial order on the set of vertices \mathcal{V} .

5.1.1. *Exact structure.* In [8], Fiebig defined a notion of exact structure in the category of \mathcal{V} as follows.

Definition 5.1. *Let $A \rightarrow B \rightarrow C$ be a sequence in \mathcal{V} . We say that it is short exact if*

$$0 \rightarrow A_{[x]} \rightarrow B_{[x]} \rightarrow C_{[x]} \rightarrow 0$$

is a short exact sequence of S -modules for any $x \in \mathcal{V}$

Remark 5.1. *Actually, this is not the original definition, which was on the whole category $\mathcal{Z}\text{-mod}^f$, but it is known to be equivalent to it if we only consider the category \mathcal{V} , that is precisely the one we are dealing with (cf.[7], Lemma 2.12).*

5.2. **Decomposition and subquotients of modules on \mathcal{Z}^J .** Lemma 5.3 describes the action of ${}^s\theta$ on the subquotients $M_{[x]}$'s. This is important in order to show that \mathcal{H}^J categorifies the parabolic Hecke algebra. Actually, to prove Lemma 5.3, we need a combinatorial result, that follows easily from the so-called lifting lemma, which we are going to recall.

Lemma 5.1 (“Lifting lemma”, cf.[14], Lemma 7.4). *Let $s \in \mathcal{S}$ and $v, u \in \mathcal{W}$ be such that $vs < v$ and $u < v$.*

- (i) *If $us < u$, then $us < vs$.*
- (ii) *If $us > u$, then $us \leq v$ and $u \leq vs$.*

Thus, in both cases, $us \leq v$.

Lemma 5.2. *Let $x \in \mathcal{W}^J$ and $t \in \mathcal{S}$. If $tx \notin \mathcal{W}^J$, then $\overline{tx} = x$.*

Proof. If $tx \notin \mathcal{W}^J$, then there exists a simple reflection $r \in J$ such that $txr < tx$ and, since $x \in \mathcal{W}^J$, $xr > x$. Using (the left version of) Lemma 5.1 (i) with $s = t$, $v = xr$ and $u = tx$, we get $txr < x$. Applying Lemma 5.1 (i) with $s = r$, $v = x$ and $u = txr$ it follows $tx > x$. Finally, from Lemma 5.1 (ii) we obtain $txr \leq x$, that, together with $x < xr$, gives $txr = x$. \square

Lemma 5.3. *Let $s \in \mathcal{S}$ and $x \in \mathcal{W}^J$, then*

$$({}^s\theta M)_{[x]} \cong \begin{cases} M_{[x]}\{-2\} \oplus M_{[sx]}\{-2\} & \text{if } sx \in \mathcal{W}^J, sx > x \\ M_{[x]} \oplus M_{[sx]} & \text{if } sx \in \mathcal{W}^J, sx < x \\ M_{[x]}\{-2\} \oplus M_{[x]} & \text{if } sx \notin \mathcal{W}^J \end{cases}$$

Proof. (cf. [11]) By Lemma 5.2, if $sx \notin \mathcal{W}^J$, then $\overline{sx} = x$ and $M_{[x]} \in {}^s\mathcal{Z}^J\text{-mod}$, so by Lemma 4.1 we get $\mathcal{Z}^J \otimes_{{}^s\mathcal{Z}^J} M_{[x]} = M_{[x]}\{-2\} \oplus M_{[x]}$.

If $x \neq sx$, we have a short exact sequence $0 \rightarrow M_{[x]} \rightarrow M_{[x,sx]} \rightarrow M_{[sx]} \rightarrow 0$ and, since ${}^s\theta$ is exact (see Corollary 4.1), ${}^s\theta M_{x,sx} = ({}^s\theta M)_{[x,sx]} = {}^s\theta M_{[x]} \oplus {}^s\theta M_{[sx]}$. Moreover ${}^s\theta M_{[x,sx]} = \mathcal{Z}^J(\{x, sx\}) \otimes_{{}^s\mathcal{Z}^J(\{x, sx\})} M_{[x,sx]}$ and the two isomorphisms follow taking in mind that $\mathcal{Z}^J(\{x, sx\})_{[x]} \cong S\{-2\}$ if $x < sx$, while $\mathcal{Z}^J(\{x, sx\})_{[x]} \cong S_k$ if $x > sx$. \square

Using induction, we get the following corollary

Corollary 5.1. *Let $M \in \mathcal{H}^J$, then for any $x \in \mathcal{W}^J$, $M_{[x]}$ is a finitely generated torsion free S -module.*

5.3. Character maps. Let A be a \mathbb{Z} -graded, free and finitely generated S -module; then $A \cong \bigoplus_{i=1}^n S\langle k_i \rangle$, for some $k_i \in \mathbb{Z}$. We can associate to A its *graded rank*, that is the following Laurent polynomial.

$$\underline{\mathrm{rk}} A := \sum_{i=1}^n v^{-k_i} \in \mathbb{Z}[v, v^{-1}].$$

This is well-defined, because the k_i 's are uniquely determined, up to the order.

Let $M \in \mathcal{H}^J$, then by Corollary 5.1, we may define a map $h^J : [\mathcal{H}^J] \rightarrow \mathbf{M}^J$ as follows.

$$h^J([M]) := \sum_{x \in \mathcal{W}^J} v^{l(x)} \underline{\mathrm{rk}} M_{[x]} H_x^{J, v^{-1}} \in \mathbf{M}^J$$

Proposition 5.1. *For each $M \in \mathcal{H}^J$ and for any $s \in \mathcal{S}$ we have $h^J([{}^s\theta M\langle 1 \rangle]) = \underline{H}_s \cdot h^J([M])$, that is the following diagram is commutative*

$$\begin{array}{ccc} [\mathcal{H}^J] & \xrightarrow{[{}^s\theta \circ \langle 1 \rangle]} & [\mathcal{H}^J] \\ h^J \downarrow & & \downarrow h^J \\ \mathbf{M}^J & \xrightarrow{\underline{H}_s} & \mathbf{M}^J \end{array}$$

Proof. (cf. [11], Proposition 4.3) By Lemma 5.3, for any $x \in \mathcal{W}^J$ we have

$$\underline{\mathrm{rk}} ({}^s\theta M)_{[x]} = \begin{cases} v^2 (\underline{\mathrm{rk}} M_{[x]} + \underline{\mathrm{rk}} M_{[sx]}) & \text{if } sx \in \mathcal{W}^J, sx > x \\ \underline{\mathrm{rk}} M_{[x]} + \underline{\mathrm{rk}} M_{[sx]} & \text{if } sx \in \mathcal{W}^J, sx < x \\ (v^2 + 1) \underline{\mathrm{rk}} M_{[x]} & \text{if } sx \notin \mathcal{W}^J \end{cases}$$

Then,

$$\begin{aligned} h^J([{}^s\theta M\langle 1 \rangle]) &= \sum_{x \in \mathcal{W}^J} v^{l(x)-1} \underline{\mathrm{rk}} ({}^s\theta M)_{[x]} H_x^{J, v^{-1}} \\ &= \sum_{\substack{x \in \mathcal{W}^J, sx \in \mathcal{W}^J \\ sx > x}} v^{l(x)+1} (\underline{\mathrm{rk}} M_{[x]} + \underline{\mathrm{rk}} M_{[sx]}) H_x^{J, v^{-1}} \\ &\quad + \sum_{\substack{x \in \mathcal{W}^J, sx \in \mathcal{W}^J \\ sx < x}} v^{l(x)-1} (\underline{\mathrm{rk}} M_{[x]} + \underline{\mathrm{rk}} M_{[sx]}) H_x^{J, v^{-1}} \\ &\quad + \sum_{x \in \mathcal{W}^J, sx \notin \mathcal{W}^J} (v^{l(x)+1} + v^{l(x)-1}) \underline{\mathrm{rk}} M_{[x]} H_x^{J, v^{-1}} \end{aligned}$$

Finally,

$$\begin{aligned} \underline{H}_s \cdot h^J([M]) &= \sum_{x \in \mathcal{W}^J} v^{l(x)} (\underline{\mathrm{rk}} M_{[x]}) \underline{H}_s \cdot H_x^{J, v^{-1}} \\ &= \sum_{\substack{x \in \mathcal{W}^J, sx \in \mathcal{W}^J \\ sx > x}} v^{l(x)} (\underline{\mathrm{rk}} M_{[x]}) (H_{sx}^{J, v^{-1}} + v H_x^{J, v^{-1}}) \\ &\quad + \sum_{\substack{x \in \mathcal{W}^J, sx \in \mathcal{W}^J \\ sx < x}} v^{l(x)} (\underline{\mathrm{rk}} M_{[x]}) (H_{sx}^{J, v^{-1}} + v^{-1} H_x^{J, v^{-1}}) \\ &\quad + \sum_{x \in \mathcal{W}^J, sx \notin \mathcal{W}^J} v^{l(x)} \underline{\mathrm{rk}} M_{[x]} (v + v^{-1}) H_x^{J, v^{-1}} \\ &= \sum_{\substack{x \in \mathcal{W}^J, sx \in \mathcal{W}^J \\ sx > x}} \left[(v^{l(x)} v \underline{\mathrm{rk}} M_{[x]}) + (v^{l(sx)} \underline{\mathrm{rk}} M_{[sx]}) \right] H_x^{J, v^{-1}} \\ &\quad + \sum_{\substack{x \in \mathcal{W}^J, sx \in \mathcal{W}^J \\ sx < x}} \left[(v^{l(x)} v^{-1} \underline{\mathrm{rk}} M_{[x]}) + (v^{l(sx)} \underline{\mathrm{rk}} M_{[sx]}) \right] H_x^{J, v^{-1}} \\ &\quad + \sum_{x \in \mathcal{W}^J, sx \notin \mathcal{W}^J} (v^{l(x)+1} + v^{l(x)-1}) \underline{\mathrm{rk}} M_{[x]} H_x^{J, v^{-1}} \\ &= h^J([{}^s\theta M\langle 1 \rangle]) \end{aligned}$$

□

6. THE CHARACTER MAP IS AN ISOMORPHISM

In order to prove that $(\mathcal{H}^J, \langle -1 \rangle, \{s\theta \circ \langle 1 \rangle\})$ is a categorification of \mathbf{M}^J , the only step left is to show that h^J is an isomorphism. The surjectivity is easy to verify, while the prove of the injectivity is rather elaborate.

Lemma 6.1. *The map $h^J : [\mathcal{H}^J] \rightarrow \mathbf{M}^J$ is surjective.*

Proof. By Theorem 2.1, $\{\underline{H}_x^{J,v^{-1}}\}_{x \in \mathcal{W}^J}$ is a basis of \mathbf{M}^J as \mathcal{L} -module. So it is enough to show that, for any $x \in \mathcal{W}^J$, there exists an object $H \in \mathcal{H}^J$ such that $h^J([H]) = \underline{H}_x^{J,v^{-1}}$. We prove the claim by induction on the length of x . By definition, $h^J(B_e) = M_e = \underline{H}_e^{J,v^{-1}}$. Now, let us consider $x \in \mathcal{W}^J$ with $l(x) = r > 0$ and a reduced expression $x = s_{i_1} \dots s_{i_r}$, with $s_{i_1}, \dots, s_{i_r} \in \mathcal{S}$. Then,

$$(7) \quad \underline{H}_x^{J,v^{-1}} = \underline{H}_{s_1}^{J,v^{-1}} \cdot \dots \cdot \underline{H}_{s_r}^{J,v^{-1}} + \sum_{\substack{y \in \mathcal{W}^J \\ y < x}} p_y \underline{H}_y^{J,v^{-1}},$$

with $p_z \in \mathbb{Z}[v, v^{-1}]$. By Proposition 5.1, we obtain

$$h^J(s_{i_1}\theta \circ \dots \circ s_{i_r}\theta B_e^J \langle n \rangle) = (\underline{H}_{s_1}^{J,v^{-1}} \cdot \dots \cdot \underline{H}_{s_r}^{J,v^{-1}})M_e = \underline{H}_{s_1}^{J,v^{-1}} \cdot \dots \cdot \underline{H}_{s_r}^{J,v^{-1}}$$

□

6.1. Injectivity. The fact that $h^\emptyset : [\mathcal{H}^\emptyset] \rightarrow \mathbf{H}$ is an isomorphism has been proven by Fiebig in [8]. We want to use this case to extend the result to any $J \subseteq \mathcal{S}$ such that \mathcal{W}_J is finite. In particular, in order to get the injectivity of h^J we will define an exact functor $I : \mathcal{H}^J \rightarrow \mathcal{H}^\emptyset$ such that the following diagram commutes:

$$\begin{array}{ccc} [\mathcal{H}^J] & \xrightarrow{[I]} & [\mathcal{H}^\emptyset] \\ h^J \downarrow & & \downarrow h^\emptyset \\ \mathbf{M}^J & \xrightarrow{i} & \mathbf{H} \end{array}$$

where $i : \mathbf{M}^J \hookrightarrow \mathbf{H}$ is the map of $\mathbb{Z}[v, v^{-1}]$ -modules given by

$$(8) \quad H_x^{J,v^{-1}} \mapsto \sum_{z \in \mathcal{W}_J} v^{l(w_J) - l(z)} H_{xz},$$

with w_J longest element of \mathcal{W}_J .

Once such a functor is given, the injectivity of the character map follows straightaway.

6.1.1. Construction of the functor I . The definition of I involves Fiebig's localisation functor \mathcal{L} (cf. [8], §3.3), which allows us to see objects of $\mathcal{Z}^J\text{-mod}$ as sheaves on the singular Bruhat moment graph \mathcal{G}^J .

The aim of this paragraph is to recall the definition the functor \mathcal{L} . Let us consider a quasi-finite \mathcal{G} . Let \mathcal{Z} be the corresponding structure algebra and $M \in \mathcal{Z}\text{-mod}^f$. For any vertex $x \in \mathcal{V}$, we set

$$(9) \quad \mathcal{L}(M)^x = M^x$$

For any edge $E : x \longrightarrow y$, let us consider $\mathcal{Z}(E) = \{(z_x, z_x) \in S \oplus S \mid z_x - z_y \in l(E)S\}$ and $M(E) := \mathcal{Z}(E) \cdot M^{x,y}$. For $m = (m_x, m_y) \in M(E)$, let us set $\pi_x((m)) = m_x$, $\pi_y((m)) = m_y$. Then

we get $\mathcal{L}(M)^E$ as the push-out in the following diagram of S -modules:

$$\begin{array}{ccc} M(E) & \xrightarrow{\pi_x} & M^x \\ \pi_y \downarrow & & \downarrow \rho_{x,E} \\ M^y & \xrightarrow{\rho_{y,E}} & \mathcal{L}(M)^E \end{array}$$

This provides us also with the restriction maps $\rho_{x,E}$ and $\rho_{y,E}$.

It is not hard to verify (cf. [8], §3.3) that this is a well-defined functor

$$(10) \quad \mathcal{L} : \mathcal{Z}\text{-mod}^f \rightarrow \mathbf{Sh}(\mathcal{G})$$

Moreover, the localisation functor \mathcal{L} turns out to be left adjoint to Γ (cf. [8], Theorem 3.5). Let us denote

- $\mathcal{Z}\text{-mod}^{\text{loc}}$ the full subcategory of $\mathcal{Z}\text{-mod}^f$, whose objects are the elements M such that there is an isomorphism $(\Gamma \circ \mathcal{L}(M)) \cong M$
- $\mathbf{Sh}(\mathcal{G})^{\text{glob}}$ the full subcategory of $\mathbf{Sh}(\mathcal{G})$, whose objects are the elements \mathcal{F} such that there is an isomorphism $(\mathcal{L} \circ \Gamma(\mathcal{F})) \cong \mathcal{F}$

Thus, the functors \mathcal{L} and Γ induce two inverses equivalences:

$$\mathcal{Z}\text{-mod}^{\text{loc}} \longleftrightarrow \mathbf{Sh}(\mathcal{G})^{\text{glob}}$$

Finally, we set $I := \langle -l(w_J) \rangle \circ \Gamma \circ p^{J,*} \circ \mathcal{L}$

6.1.2. I preserves special modules. In the previous paragraph, we defined a functor $I : \mathcal{Z}^J\text{-mod}^f \rightarrow \mathcal{Z}\text{-mod}^f$. We want to show now that it actually maps \mathcal{H}^J to \mathcal{H} . In order to do this, we need to recall the moment graph analogue of a theorem by Deodhar relating parabolic Kazhdan-Lusztig polynomials and regular ones.

Theorem 6.1 ([18]). *Let $J \subseteq \mathcal{S}$ be such that W_J is finite, with longest element w_J . Let $w, x \in \mathcal{W}^J$, then $p^{J,*}(\mathcal{B}^J(w)) \cong \mathcal{B}(ww_J)$ as sheaves on $\mathcal{G} = \mathcal{G}(\mathcal{W}, \emptyset)$.*

Proposition 6.1 will allow us to see any element in \mathcal{H}^J as the space of global sections of some BMP-sheaf on \mathcal{G}^J . From now on, we will denote by $B^J(w)$ the space of global sections of the indecomposable BMP-sheaf $\mathcal{B}^J(w) \in \mathbf{Sh}(\mathcal{G}^J)$. Let us recall a fundamental characterisation of $B^J(w)$.

Theorem 6.2 (cf. [7], Theorem 5.2.). *For any $w \in \mathcal{G}^J$ $B^J(w) \in \mathcal{V}$ is indecomposable and projective. Moreover, every indecomposable projective object in \mathcal{V} is isomorphic to $B^J(w)$ for a unique $w \in \mathcal{G}^J$.*

Proposition 6.1. *$M \in \mathcal{H}^J$ if and only there exists a BMP-sheaf $\mathcal{B} \in \mathbf{Sh}(\mathcal{G}^J)$ such that $M \cong \Gamma(\mathcal{B})$ as \mathcal{Z}^J -modules.*

Proof. By induction, from the exactness of ${}^s\theta^J$, it follows that the objects of \mathcal{H}^J are all projective and then, by Theorem 6.1 6.2, any $M \in \mathcal{H}^J$ may be identified with the space of global sections of a BMP-sheaf on \mathcal{G}^J .

We want now to show that, for any $x \in \mathcal{W}_J$, $B^J(x) \in \mathcal{H}^J$. We prove the claim by induction on $\sharp\text{supp}(M)$, where $\text{supp}(M) = \{x \in \mathcal{W}^J \mid M^x \neq 0\}$. Clearly, $B_e \cong B^J(e)$.

The statement follows straightforwardly, once proved that, if $sx > x$, then ${}^s\theta^J(B^J(x)) = B^J(sx) \oplus B$.

At first we show that $\text{supp}({}^s\theta^J(B^J(x))) \subseteq \{\leq sx\}$, that is $({}^s\theta^J(B^J(x)))^y = 0$ for all $y \notin \{\leq sx\} \cap \mathcal{W}^J$. From Lemma 5.3, it follows easily that $({}^s\theta^J(B^J(x)))_{[y]} = 0$ for all $y \notin \{\leq sx\} \cap \mathcal{W}^J$.

Let us observe that, as ${}^s\theta^J(B^J(x)) \in \mathcal{H}^J$, from what we have proved above, there exist $w_1, \dots, w_r \in \mathcal{W}^J$ and k_1, \dots, k_r such that ${}^s\theta^J(B^J(x)) = \oplus_{i=1}^r B^J(w_i)\langle k_i \rangle$ and, for any $y \in \mathcal{W}^J$,

$$(\oplus_{i=1}^r B^J(w_i)\langle k_i \rangle)_{[y]} = \oplus_{i=1}^r B^J(w_i)_{[y]}\langle k_i \rangle$$

So, in particular, for all $y \notin \{\leq sx\} \cap \mathcal{W}^J$,

$$\begin{aligned} 0 &= B^J(w_i)_{[y]} \\ &= \ker(\rho_{\delta y} : \mathcal{B}^J(w_i)^y \rightarrow \mathcal{B}^J(w_i)^{\delta y}) \end{aligned}$$

This implies $\mathcal{B}^J(w_i)^y = B^J(w_i)^y = 0$ for all $i = 1, \dots, r$, and so

$${}^s\theta^J(B^J(x)) = \oplus_{i=1}^r B^J(w_i)\langle k_i \rangle$$

where $w_i \in \{\leq sx\}$ for all $i = 1, \dots, r$.

Now is left to show that there exists at least one $i \in \{1, \dots, r\}$ such that $w_i = sx$. By applying once again Lemma 5.3, we get $({}^s\theta^J(B^J(x)))^{sx} = ({}^s\theta^J(B^J(x)))_{[sx]} \cong S$ and hence the statement. \square

Corollary 6.1. *The functor I maps \mathcal{H}^J to \mathcal{H} .*

Proof. Let $M \in \mathcal{H}^J$, then, by Proposition 6.1, there exist $w_1, \dots, w_r \in \mathcal{W}^J$ and $m_1, \dots, m_r \in \mathbb{Z}$ such that $M = \bigoplus_{i=1}^r B^J(w_i)\langle m_i \rangle$. Then, we get the following.

$$\begin{aligned} I(M) &= I(\bigoplus_{i=1}^r B^J(w_i)\langle m_i \rangle) \\ &= \bigoplus \Gamma \circ p^{J,*} \circ \mathcal{L}(B(w_i w_J))\langle m_i - l(w_J) \rangle \end{aligned}$$

\square

6.1.3. *Injectivity and exactness of I .*

Lemma 6.2. *Let $w \in \mathcal{W}^J$ and let w_J be the longest element of \mathcal{W}_J . Then, for all $x \in \mathcal{W}$,*

$$(p^{J,*} \mathcal{B}^J(w_J))_{[x]} = \left(\prod_{\substack{y \in \mathcal{V}^{\delta x} \\ y \in x \mathcal{W}_J}} \alpha_y \right) \mathcal{B}^J(w)_{[x^J]}.$$

Proof. For $z \in \mathcal{W}^J$ and E edge of $\mathcal{G}^J = \mathcal{G}(\mathcal{W}, J)$, let us denote by $\rho_{z,E}$ the corresponding restriction map. Then, we have the following.

$$\begin{aligned} (p^{J,*} \mathcal{B}^J(w_J))_{[x]} &= \bigcap_{y \in \mathcal{V}^{\delta x}} \ker((p^{*,J} \rho)_{x,x \rightarrow y}) \\ &= \left(\bigcap_{\substack{y \in \mathcal{V}^{\delta x} \\ y \notin x \mathcal{W}_J}} \ker(\rho_{x^J, x^J \rightarrow y^J}) \right) \cap \left(\bigcap_{\substack{y \in \mathcal{V}^{\delta x} \\ y \in x \mathcal{W}_J}} \ker \pi_{x,x \rightarrow y} \right) \end{aligned}$$

where $\pi_{x,x \rightarrow y} : \mathcal{B}^J(w)^{x^J} \rightarrow \mathcal{B}^J(w)^{x^J} / \alpha_y \mathcal{B}^J(w)^{x^J}$ is the canonical quotient map and α_y is a generator of $l(x \rightarrow y)$.

Let us observe that, by definition,

$$\bigcap_{\substack{y \in \mathcal{V}^{\delta x} \\ y \notin x \mathcal{W}_J}} \ker(\rho_{x^J, x^J \rightarrow y^J}) = \mathcal{B}^J(w_J)_{[x]}$$

Moreover, since there is at most one edge adjacent to x labelled by a multiple of α_y , the labels of such edges are pairwise linearly independent and we get

$$\bigcap_{\substack{y \in \mathcal{V}^{\delta x}, \\ y \in x\mathcal{W}_J}} \ker \pi_{x, x \rightarrow y} = \prod_{\substack{y \in \mathcal{V}^{\delta x}, \\ y \in x\mathcal{W}_J}} \alpha_y \mathcal{B}^J(w)^{x^J}$$

It follows,

$$(p^{J,*} \mathcal{B}^J(ww_J))_{[x]} = \left(\prod_{\substack{y \in \mathcal{V}^{\delta x}, \\ y \in x\mathcal{W}_J}} \alpha_y \right) \mathcal{B}^J(w)_{[x^J]}$$

This concludes the proof of the lemma. \square

Corollary 6.2. *The functor I induces an injective map $[I] : [\mathcal{H}^J] \hookrightarrow [\mathcal{H}]$.*

Proof. Let $M \in \mathcal{H}^J$ and let us suppose $[I]([M]) = [I(M)] = 0$. It follows $\underline{\text{rk}}(I(M))_{[x]} = 0$ for all $x \in \mathcal{W}$, that is $(I(M))_{[x]} = 0$ for all $x \in \mathcal{W}$.

Assume that $[M] \neq 0$, then there exist $w_1, \dots, w_r \in \mathcal{W}^J$ and $n_1 \dots n_r$ such that $M = \bigoplus_{i=1}^r B^J(w_i) \langle n_i \rangle$. Since $(I(\bigoplus_{i=1}^r B^J(w_i) \langle n_i \rangle))_{[x]} = \bigoplus_{i=1}^r I(B^J(w_i))_{[x]} \langle n_i \rangle$, we may suppose $M = B^J(w)$ for some $w \in \mathcal{W}^J$. Then, for any $x \in \mathcal{W}$, we get the following.

$$\begin{aligned} 0 &= I(B^J(w))_{[x]} \\ &= \left(\prod_{\substack{y \in \mathcal{V}^{\delta x}, \\ y \in x\mathcal{W}_J}} \alpha_y \right) \cdot B^J(w)_{[x^J]} \langle -l(w_J) \rangle \end{aligned}$$

It follows $B^J(w)_{[x^J]} = 0$ for all $x^J \in \mathcal{W}^J$, since $B^J(w)_{[x^J]}$ is a free S -module for any $x^J \in \mathcal{W}^J$. In particular, we obtain $B^J(w)_{[w]} = B^J(w)^w = 0$, but this is not possible. \square

Proposition 6.2. *The functor I is exact, with respect to the exact structure in §5.1.1.*

Proof. Let us take $M, N \in \mathcal{H}^J$, with $M = \bigoplus_{i \in I} B^J(w_i) \langle m_i \rangle$ and $N = \bigoplus_{j \in J} B^J(w_j) \langle n_j \rangle$

Let us consider the map $f : L \rightarrow M$ and the induced maps $f_{[x^J]} : M_{[x^J]} \rightarrow N_{[x^J]}$ for any $x^J \in \mathcal{W}^J$. Now, thanks to Lemma 6.2, it is easy to describe $I(f)_{[x]}$. Namely, if $\prod_{\substack{y \in \mathcal{V}^{\delta x}, \\ y \in x\mathcal{W}_J}} \alpha_y = \alpha_{i_1} \cdot \dots \cdot \alpha_{i_r}$, we obtain

$$\begin{aligned} I(f) : \quad & I(M)_{[x]} \longrightarrow I(N)_{[x]} \\ & (\alpha_{i_1} \cdot \dots \cdot \alpha_{i_r})m \longmapsto (\alpha_{i_1} \cdot \dots \cdot \alpha_{i_r})f_{[x]}(m) \end{aligned}$$

Now it is clear that, if $0 \rightarrow L_{[x]} \rightarrow M_{[x]} \rightarrow N_{[x]} \rightarrow 0$ is a short exact sequence of S -modules, then $0 \rightarrow (IL)_{[x]} \rightarrow (IM)_{[x]} \rightarrow (IN)_{[x]} \rightarrow 0$ is also exact. \square

6.1.4. Commutativity of the diagram. The last step missing is the commutativity of Diagram 6.3. Before proving it, we need the following preliminary lemma.

Lemma 6.3. *There is an isomorphism $B^\emptyset(ww_J)_{[x]} \cong B^J(w)_{[x^J]} \langle 2l(x_J) - 2l(w_J) \rangle$ of graded S -modules.*

Proof. By Theorem 6.1, $\mathcal{B}^\emptyset(w_J) \cong p^{J,*} \mathcal{B}^J(w_J)$ as sheaves on $\mathcal{G} = \mathcal{G}(\mathcal{W}, \emptyset)$. It follows that, for any $x \in \mathcal{W}$, $\mathcal{B}^\emptyset(w_J)_{[x]} \cong (p^{J,*} \mathcal{B}^J(w_J))_{[x]}$ as graded S -modules and then, by Lemma 6.2, we get

$$\begin{aligned} B^\emptyset(w_J)_{[x]} &\cong \left(\prod_{\substack{y \in \mathcal{V}^{\delta x} \\ y \in x\mathcal{W}_J}} \alpha_y \right) B^J(w)_{[x^J]} \\ &\cong B^J(w)_{[x^J]} \langle 2 \cdot \sharp\{y \in \mathcal{V}^{\delta x}, y \in x\mathcal{W}_J\} \rangle \end{aligned}$$

Now, if \mathcal{T}_J is the set of reflections of \mathcal{W}_J ,

$$\begin{aligned} \sharp\{y \in \mathcal{V}^{\delta x}, y \in x\mathcal{W}_J\} &= \sharp\{z \in \mathcal{W}_J \mid \exists t \in \mathcal{T}_J : z = tx_J \text{ and } x_J < z\} \\ &= l(w_J) - l(x_J) \end{aligned}$$

□

Finally, we are able to prove the following proposition, which concludes the proof of Theorem 4.1.

Proposition 6.3. *The following diagram is commutative*

$$\begin{array}{ccc} [\mathcal{H}^J] & \xrightarrow{[I]} & [\mathcal{H}^\emptyset] \\ h^J \downarrow & & \downarrow h^\emptyset \\ \mathbf{M}^J & \xrightarrow{i} & \mathbf{H} \end{array}$$

Proof. As $I(\bigoplus_{i \in I} B^J(w_i)) = \bigoplus I(B^J(w_i))$, it is enough to prove the statement for an indecomposable projective $B^J(w)$.

$$\begin{aligned} I(B^J(w)) &= \langle -l(w_J) \rangle \circ \Gamma \circ p^{J,*} \circ \mathcal{L}(B^J(w)) \\ &= \langle -l(w_J) \rangle \circ \Gamma \circ p^{J,*}(\mathcal{B}^J(w)) \\ &\cong \langle -l(w_J) \rangle \circ \Gamma(\mathcal{B}(ww_J)) \\ &= B(ww_J) \langle -l(w_J) \rangle \end{aligned}$$

Thus, if $B^J(w)_{[x^J]} = \bigoplus_{i \in I_{x^J}} S\langle k_i \rangle$, we get

$$\begin{aligned} h^\emptyset \circ [I]([B^J(w)]) &= h^\emptyset(B^\emptyset(ww_J) \langle l(w_J) \rangle) \\ &= \sum_{x \in \mathcal{W}} v^{-l(w_J) + l(x)} \underline{\text{rk}} B^\emptyset(ww_J)_{[x]} H_x \\ \text{(by Lemma 6.3)} &= \sum_{x \in \mathcal{W}} v^{l(w_J) + l(x)} \underline{\text{rk}} (B^J(w)_{[x^J]} \langle 2l(x_J) - 2l(w_J) \rangle) H_x \\ &= \sum_{x \in \mathcal{W}} v^{-l(w_J) + l(x)} \left(\sum_{i \in I_{x^J}} v^{-2l(x_J) + 2l(w_J) - k_i} \right) H_x \\ &= \sum_{x \in \mathcal{W}} v^{l(w_J) + l(x)} \left(\sum_{i \in I_{x^J}} v^{-2l(x_J) - k_i} \right) H_x \end{aligned}$$

where $H_x = H_x^{\emptyset, v^{-1}}$. On the other hand, we have the following.

$$\begin{aligned} i \circ h^J([B^J(w)]) &= i \left(\sum_{x^J \in \mathcal{W}^J} v^{l(x^J)} \underline{\text{rk}} B^J(w)_{[x^J]} H_{x^J}^{J, v^{-1}} \right) \\ &= \sum_{x^J \in \mathcal{W}^J} \left[v^{l(x^J)} \left(\sum_{i \in I_{x^J}} v^{-k_i} \right) i(H_{x^J}^{J, v^{-1}}) \right] \\ &= \sum_{x^J \in \mathcal{W}^J} \left[v^{l(x^J)} \left(\sum_{i \in I_{x^J}} v^{-k_i} \right) \left(\sum_{x_J \in \mathcal{W}_J} v^{l(w_J) - l(x_J)} H_{x^J x_J} \right) \right] \\ &= \sum_{x^J \in \mathcal{W}^J} \sum_{x_J \in \mathcal{W}_J} \left(\sum_{i \in I_{x^J}} v^{l(x^J) - k_i + l(w_J) - l(x_J)} \right) H_{x^J x_J} \\ &= \sum_{x \in \mathcal{W}} v^{l(w_J) + l(x)} \left(\sum_{i \in I_{x^J}} v^{-2l(x_J) - k_i} \right) H_x \end{aligned}$$

□

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